ICB
Institut für Informatik und
Wirtschaftsinformatik

Freimut Brenner

Cumulative Measures of Absorbing Joint Markov Chains and an Application to Markovian Process Algebras

12

ICB-RESEARCH REPORT

ICB-Research Report No.12
February 2007
Universität Duisburg-Essen

Authors' Address:

Freimut Brenner

Institut für Informatik und
Wirtschaftsinformatik (ICB)
Universität Duisburg-Essen
D-45117 Essen

Freimut.Brenner@icb.uni-due.de

Managing Assistant and Contact:

Jürgen Jung

Institut für Informatik und
Wirtschaftsinformatik (ICB)
Universität Duisburg-Essen
Universitätsstr. 9
45141 Essen
Germany

Email: icb@uni-duisburg-essen.de

ISSN 1860-2770

Preleminary Remark

The work presented in this note is planned to evolve into the theoretical core of the author's doctoral thesis and should be regarded as work in progress. Objections and suggestions are welcome.

Abstract

Markov Models are of outstanding importance in the performance and reliability evaluation of computer systems and communication networks. In this paper we aim at contributing to the field of Markovian Process Algebras (MPAs). An MPA model is (or may be) the composition of several concurrent sub-components (each of which describes an underlying Markov chain) which may interact with each other through synchronisation. On the one hand the existence of sub-components implies the possibility of the state space explosion problem, i.e. the size of the state space of the Markov chain underlying the composite component grows exponentially in the number of sub-components. On the other hand the interaction of sub-components in general negates the property of independence of their underlying Markov chains, and hence forbids a product-form solution for steady state probabilities.

Our target quantities are single steady state probabilities of the Markov chain underlying the composite component. We consider composite components which possess only global synchronisations, i.e. every sub-component is involved in every synchronisation. For this class of MPA models the behaviour of the composite component between two successive synchronisations can be described by the joint process of several absorbing Markov chains.

First, a new result on cumulative measures of absorbing joint Markov chains is presented. We compute the mean time to absorption and the mean time the joint Markov chain spends in a certain set before absorption. Our computations do not operate on the state space of the joint Markov chain, and hence the problem of state space explosion is avoided. The computational effort of our method rather depends on convergence properties of the joint Markov chain.

Afterwards, this result is applied to compute steady state probabilities for a class of composite components specified as PEPA models which are popular ambassadors of MPAs. It is easily understood that these results carry over from PEPA to other MPA variants.

Contents

1 Introduction

The analysis of stochastic systems is a branch of applied research which spreads across many disciplines of science. Areas of application include performance and reliability of computing systems and telecommunication networks, and also fields like reaction kinetics in physical chemistry and financial risk theory, to name but a few.

Often a model of such a stochastic system is forced to fit into a Markovian framework, i.e. a Markov chain can be extracted from that model. As a consequence the model can be analysed by means provided by the rich and often times elegant theory of Markov chains. In addition, formalisms to build (or describe) Markovian models exist. These formalisms constitute the advantage that they (may or may not) equip the model, or certain activities or states of that model, with an intuitive meaning. Queueing stations (and networks), stochastic Petri nets and Markovian process algebras are outstanding examples for formalisms which have shown their usefulness in the areas of performance and reliability evaluation of computer and communication networks. For an extended overview and application examples see e.g. [BGMT98], [CaTu02], [HLR00] and [BHK00], or the latest proceedings of the conferences MMB ([GeHe06]), QEST ([AMR06]) and Performance ([MKS06]).

When dealing with Markov chain formalisms, the first two questions should be: a) Is that formalism useful from the modellers point of view? and b) Can the special structure which the formalism induces on the underlying Markov chain be exploited to derive efficient (or elegant) solutions of that Markov chain? For instance, the solution of a birth-death process is given by a simple symbolic expression; BCMP networks possess a product-form solution ([BCMP75]); nearly completely decomposable Markov chains can be partitioned into suited subchains, where transitions inside of subchains and transitions between subchains can be treated separately ([Cour77]).

In this paper we aim at contributing to the field of Markovian Process Algebras (MPAs). An MPA model consists of several concurrent components which may interact with each other through synchronisation. On the one hand the existence of concurrent components imply the possibility of the state space explosion problem, and on the other hand the interaction of components in general negates the property of independence of components, and hence forbids a product-form solution. We will make use of the fact that for a certain class of MPA models the concurrent components behave independently of each other between successive points of synchronisation.

The paper is organised as follows: In chapter 2 of this paper we present a new result on absorbing continuous time Markov chains which are built of several marginal absorbing Markov chains. This result is applied to compute steady state probabilities for a class of PEPA components in chapter 3 (PEPA is a popular variant of an MPA).

ad chapter 2: For a Markov chain $U = (U_1, \ldots, U_m)$, with the marginal absorbing Markov chains U_i, $i = 1 \ldots m$, we propose a new method to compute:

(a) The mean time to absorption.

(b) The mean time which the Markov chain U spends in some set A before absorption. The set A is restricted to possess the form $A = \times_{i=1}^{m} A_i$, where the A_i are subsets of the marginal state spaces.

Unlike classical approaches to compute these two quantities our method does not carry out operations on the global state space, i.e. on the state space of U. Note, that the global state space grows exponentially in the number m of involved marginal processes.

Without diving into detail at this point, we explain the basic idea of our method. Let $\mathsf{A} = \times_{i=1}^{m} \mathsf{A}_i$ be a subset of the state space of U, where the A_i are subsets of the marginal state spaces. The key will be to reformulate the probabilities $\nu(n)[\mathsf{A}]$, $n \geq 0$, that the discrete time Markov chain (DTMC) embedded in U – more precisely this DTMC is obtained from U by uniformisation – is in some set A in the $n-$th step. At first the marginal CTMCs are uniformised, which yields embedded discrete time Markov chains. These DTMCs are then solved in isolation. In this context the solution is considered the discrete function $\nu_i(\cdot)[\mathsf{A}_i]$, where for fixed i and A_i the quantity $\nu_i(n)[\mathsf{A}_i]$ is the probability that the DTMC embedded in U_i is in set A_i in the $n-$th step. By a convolution-like operator $\star$ these solutions are combined to the solution of the DTMC embedded in U. That means we obtain a function $\nu(\cdot)[\mathsf{A}]$, where on the one hand $\nu(n)[\mathsf{A}] = (\nu_1[\mathsf{A}_i] \star \cdots \star \nu_m[\mathsf{A}_m])(n)$ and on the other hand $\nu(n)[\mathsf{A}]$ is the probability that the DTMC embedded in the joint Markov chain U is in set A in the $n-$th step. Provided that the expected values in (a) and (b) exist, they can be expressed by infinite and converging series involving the functions $\nu[S]$ (i.e. take $\mathsf{A} := S$, set of absorbing states of U) and $\nu[A]$ (i.e. take $\mathsf{A} := A$).

The method we propose is an exact method, where, of course, the computation of the series just mentioned requires truncation at a certain index. The time complexity is $O(Nmd^2 + mN^2)$, where m is the number of marginal CTMCs, d is the maximal size of the marginal state spaces and N is the truncation index of the infinite series which depends on convergence properties (eigenvalues) of the joint CTMC and the desired accuracy.

ad chapter 3: Stochastic process algebras have become popular since the formalism was proposed by Herzog in [Herz90]. In particular, Markovian Process Algebras (MPAs) have drawn much attention due to the the fact that the quantitative solution of an MPA happens to be the solution of the underlying Markov chain. Examples for MPAs involve PEPA ([Hill96]), EMPA ([BDG94], [BeGo96]), MTIPP ([HeRe02]) and IMC ([Herm02]).

On the one hand MPAs allow to define components in isolation whose behaviour is determined by underlying Markov chains. On the other hand these components can be combined to a composite component by some cooperation operator. This operation forces components to interact or cooperate with each other via synchronisation. Basically, all components act independently of each other until they reach some point, where they are forced to synchronise with other components. Components that are ready to synchronise must wait for the other involved components to become ready on their part. When all processes have reached a point where they are ready to synchronise the synchronisation is executed and afterwards the components again evolve independently of each other. Depending on the specific MPA the synchronisation itself can possess an exponentially distributed duration or it can be timeless. In either case all synchronising components begin to synchronise at a common time instant and they end the synchronisation at a common time instant (here, we interpreted a timeless synchronisation to possess the duration 0).

Obviously, waiting times (until synchronisation can take place) of components depend on the behaviour of other components, and hence synchronising components are not independent of each other. In general, this permits product-form solutions for the Markov chain underlying the composite component. Since the state space of this Markov chain grows exponentially with the number of sub-components, the computation of quantitative measures is subject to the state space explosion problem. An overview of different methods which tackle this challenge can be found in [Hill99].

In his dissertation [Bohn02] Bohnenkamp took on a view on MPAs which had not been investigated up to that point. He considered a special class of MPA components where there exist only global synchronisations, i.e. every sub-component must participate in every synchronisation. Then, points of synchronisation define an embedded DTMC of the composite component. Together with the first passage times from embedded states to their embedded successor states the embedded DTMC defines an embedded semi-Markov chain. Solving the embedded DTMC and computing the expected values of these first passage times yields the steady state distribution of the semi-Markov chain. If in addition some more computations are carried out on local components, **local** steady state probabilities can be determined.

In [Bohn02] Bohnenkamp exploited the fact that such a first passage time is the maximum of, say m, phase-type distributions. In other words, it is the mean value of the maximum of the times to absorption of m absorbing Markov chains. An algorithm to compute this mean time is given in the cited work.

In chapter 3 we take on the view of Bohnenkamp and apply it to PEPA. We realise that the maximum of the times to absorption of m absorbing Markov chains is just the time to absorption of the joint CTMC of these absorbing Markov chains. We compute the mean time to absorption of the joint CTMC, and **in addition** we will also compute the mean time the joint CTMC spends in some set A before absorption. For these computations we employ the method developed in chapter 2, and hence we never operate on the global state space of the composite PEPA component. Due to computing these additional quantities, we will be able to derive the **global** steady state probability that the composite PEPA component is in set A.

In chapter 4 we conclude.

2 Cumulative Measures of Absorbing Joint Markov Chains

This chapter deals with cumulative measures of an absorbing Markov chain U which is the joint process of m marginal absorbing CTMCs U_i, $i = 1 \dots m$. In this context the mean time to absorption and the mean time spent in some set A before absorption will be of particular interest to us.

We set out with some basic properties of absorbing joint Markov chains in section 2.1. Methods that compute the two quantities mentioned above are briefly discussed. Furthermore, basic convergence properties of the joint process U and the strongly related topic of eigenvalues are addressed.

Sections 2.2 and 2.3 contain the actual accomplishment of this chapter. At first we show how transient probabilities of a discrete time Markov chain embedded in U can be obtained in a compositional way by transient probabilities of DTMCs embedded[1] in the marginal CTMCs U_i, $i = 1 \dots m$, and hence the global state space (i.e. the state space of U) needs not to be constructed nor being operated on. This results in formulas for the computation of the desired two mean values (the mean time to absorption and the mean time spent in A before absorption) whose computation time does not depend on the size of the global state space. The computation times of these formulas depend on the number of marginal CTMCs, the sizes of the marginal state spaces and the convergence speed of the joint CTMC U. The algorithm which computes the two mean values is the topic of section 2.3.

Finally, a list containing most of the notation used throughout this chapter can be found in section 2.4.

[1]The embedded DTMCs refer to uniformised variants of U and the U_i, $i = 1 \dots m$.

2.1 Basic Properties of Absorbing Joint Markov Chains

Let $U_i = (U_i(t))_{t\in\mathbb{R}_{\geq 0}}$, $i = 1,\ldots,m$, be m independent homogeneous absorbing continuous time Markov chains, where U_i, $1 \leq i \leq m$, is defined by the finite state space E_i, the starting state $s'_i \in E_i$, the set of absorbing states $S_i \subset E_i$,and the generator matrix $Q_i = (Q_i(j,\ell))_{j,\ell\in E_i}$.

Now define the Markov chain U as the joint process of $U_1,\ldots,U_m$, i.e. $U = (U(t))_{t\in\mathbb{R}_{\geq 0}} := (U_1,\ldots,U_m)$. Then the state space of U is given by $E = \times_{i=1}^m E_i$, the starting state of U is $s' = (s'_1,\ldots,s'_m)$ and the set of absorbing states is given by $S = \times_{i=1}^m S_i$. It is well-known that the generator matrix Q of the joint CTMC U can be represented by the Kronecker sum of the generator matrices of the marginal CTMCs, i.e.

$$Q = \oplus_{i=1}^m Q_i. \tag{2.1}$$

The transient probability distribution $p(t)$ of U at time t is then given by the matrix exponential

$$p(t) = p(0)e^{Qt}, \tag{2.2}$$

where $p(0)$ denotes the initial distribution of U. Alternatively, the distribution $p(t)$ can be expressed as

$$p(t) = \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{-qt} p(0)P^n, \tag{2.3}$$

where $P = I + \frac{1}{q}Q$ for some $q \geq \max_{k\in E}\{|Q(k,k)|\}$. Defining $\nu(0) := p(0)$ and $\nu(n) = \nu(n-1)P = p(0)P^n$, for $n > 0$, we also have

$$p(t) = \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{-qt} \nu(n). \tag{2.4}$$

Algorithms building on equation (2.3) or (2.4) to compute transient distributions are widely known as uniformisation method or Jensen's method. Note, that it is recommended to choose $q > \max_{k\in E}\{|Q(k,k)|\}$, in order to avoid periodicities.

Assumptions Throughout This Paper. Throughout this paper we impose the following restrictions on the CTMC U.

- The marginal state spaces E_i, $i = 1\ldots m$, are finite. This implies that the state space E of U is finite.
- $P = I + \frac{1}{q}Q$ is aperiodic. Note, that $q > \max_{k\in E}\{|Q(k,k)|\}$ implies aperiodicity.
- All states not contained in S are transient. That means U possesses exactly $|S|$ recurrent classes, where each absorbing state forms a recurrent class.

2.1.1 Convergence Speed and Eigenvalues

Eigenvalues of P. The following properties about the eigenvalues of the matrix P are well-known (see e.g. [Stew94]).

- P possesses $dim(P)$ eigenvalues, if counting multiplicities.
- P is a stochastic matrix, and hence the multiplicity of the eigenvalue 1 equals the number of recurrent classes of U, i.e. the number of absorbing states $|S|$.
- Since P is aperiodic, there are no other eigenvalues than 1 with modulus 1.

In the following assume that P possesses d distinct eigenvalues, i.e. we are not counting multiplicities, and assume the following indexing

$$1 = |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_d|. \tag{2.5}$$

Convergence Speed of U. It is a well-known fact that the convergence speed of a discrete time Markov chain is strongly connected with the eigenvalues of the transition matrix P. To understand this we consider the following iterative procedure that computes the stationary distribution π of this Markov chain.

$$\nu(k) = \nu(k-1)P, \quad k \geq 1, \tag{2.6}$$

$$\lim_{k\to\infty} \nu(k) \to \pi, \tag{2.7}$$

where $\nu(0)$ is the initial probability distribution. We see that one iteration step of (2.6) is exactly one step of the power method to compute the eigenpair (λ_1, π) of P, where λ_1 is the largest eigenvalue in modulus of P and π is the corresponding left eigenvector. Of course $\lambda_1 = 1$, since P is stochastic and π is a stationary distribution of the discrete time Markov chain.

The speed of convergence of (2.6) is connected to the eigenvalue λ_2 by

$$\parallel \nu(k) - \pi \parallel = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right). \tag{2.8}$$

Hence, there exists a positive constant c', with

$$\lim_{k\to\infty} \parallel \nu(k) - \pi \parallel = c' \cdot |\lambda_2|^k. \tag{2.9}$$

From this follows that there exists also a positive constant c'', such that with the set of absorbing states S

$$\lim_{k\to\infty} |\nu(k)(S) - \pi(S)| = c'' \cdot |\lambda_2|^k. \tag{2.10}$$

With $\pi(S) = 1$, $\nu(k)(S) \leq 1$, this can be written as

$$\lim_{k\to\infty} \frac{1 - \nu(k)(S)}{|\lambda_2|^k} = c''. \tag{2.11}$$

This implies the important relation

$$\lim_{k\to\infty} \frac{1 - \nu(k+1)(S)}{1 - \nu(k)(S)} = |\lambda_2|. \tag{2.12}$$

Provided that the values $\nu(k)(S)$ are known up to some suited index K, $|\lambda_2|$ and the constant c'' can be estimated by exploiting equations (2.12) and (2.11). Of course, for a suited constant $c > c''$ the following bound can be given

$$1 - \nu(k)(S) \leq c \cdot |\lambda_2|^k, \quad \text{for } k > K. \tag{2.13}$$

Eigenvalues and the Marginal CTMCs. In the preceding paragraph we have seen that the second largest eigenvalue modulus $|\lambda_2|$ of P can be estimated if the quantities $\nu(n)(S)$ are known up to a certain index K. Here, we briefly point out how this eigenvalue modulus can be bounded in terms of eigenvalues of the marginal generator matrices Q_i, $i = 1 \ldots m$.

Corollary 1. *For the generator matrices Q_i, $i = 1 \dots m$, and $Q = \oplus_{i=1}^m Q_i$ let $P = I + 1/qQ$ and $P_i = I + 1/q_i Q_i$, where $q_i \geq max_j\{|Q_i(j,j)|\}$, $i = 1 \dots m$, and $q \geq max_j\{|Q(j,j)|\}$. Assume that P and the P_i, $i = 1 \dots m$, are aperiodic. Then the following is true.*

(U.1) λ is an eigenvalue of $Q \iff \exists\, \lambda^{(i)}, i = 1 \dots m,$ with $\lambda^{(i)}$ eigenvalue of Q_i and $\lambda = \sum_i \lambda^{(i)}$.

(U.2) λ is an eigenvalue of $Q \iff \lambda' := \lambda/q + 1$ is an eigenvalue of P. This result also holds if replacing Q, P and q by Q_i, P_i and q_i.

(U.3) $\exists\, \lambda^{(i)}, i = 1 \dots m,$ with $\lambda^{(i)}$ eigenvalue of Q_i and $\lambda' = 1 + 1/q \sum_i \lambda^{(i)} \iff \lambda'$ is an eigenvalue of P.

(U.4) Q and Q_i, $i = 1 \dots m$, possess the eigenvalue 0.

(U.5) Every eigenvalue $\lambda^{(i)} \neq 0$ of Q_i lies in the complex plane within a circle with center $-q_i$ and radius q_i.

(U.6) Every eigenvalue $\lambda^{(i)}$ of Q_i satisfies $|q_i + \lambda^{(i)}| \leq q_i$. If $\lambda^{(i)} \neq 0$ then we have the strict inequality $|q_i + \lambda^{(i)}| < q_i$

Proof. (U.1) is a basic property of the Kronecker sum by which Q can be represented (cf. equation (2.1)). (U.2) is a direct consequence of $P = I + \frac{1}{q}Q$. (U.3) follows by combining (U.1) and (U.2). (U.4) results from Q and the Q_i, $i = 1 \dots m$, being singular matrices. (U.5) follows from the fact that every eigenvalue $1 + 1/q_i\lambda^{(i)} \neq 1$ of the stochastic and aperiodic matrix $P_i := I + 1/q_i Q_i$ lies within the unit circle in the complex plane, and hence transforming P_i into Q_i implicates that the eigenvalues of P_i are shifted to the left by 1 and afterwards scaled by the factor q_i. (U.6) follows from (U.5) by noting that for every eigenvalue $\lambda^{(i)} \neq 0$ of Q_i the number $q_i + \lambda^{(i)}$ lies in the complex plane within a circle with center 0 and radius q_i.

□

The following corollary bounds the second eigenvalue modulus $|\lambda_2|$ of P in terms of eigenvalues of the Q_i, $i = 1 \dots m$.

Corollary 2. *For the generator matrices Q_i, $i = 1 \dots m$, and $Q = \oplus_{i=1}^m Q_i$ let $P = I + 1/qQ$, where $q_i \geq max_j\{|Q_i(j,j)|\}$, $i = 1 \dots m$, and $q = q_1 + \dots + q_m$. Assume that P is aperiodic. Let Λ_i be the set of distinct eigenvalues of Q_i, $i = 1 \dots m$, and define the values max and $i(max)$ as*

$$max = \max_{\substack{1 \leq i \leq m \\ \lambda^{(i)} \in \Lambda_i \setminus \{0\}}} \{|q_i + \lambda^{(i)}|\} \quad \text{and} \quad i(max) = i \iff max = |q_i + \lambda^{(i)}|. \tag{2.14}$$

Then the second largest eigenvalue modulus of P satisfies

$$|\lambda_2| \leq \frac{1}{q}\left(max + \sum_{\substack{1 \leq i \leq m \\ i \neq i(max)}} q_i \right). \tag{2.15}$$

Proof. First note that by the representation of Q as the Kronecker sum of the Q_i, from $q_i \geq max_j\{|Q_i(j,j)|\}$, $i = 1 \dots m$, follows $q = q_1 + \dots + q_m \geq max_j\{|Q(j,j)|\}$. Thus, P is indeed a stochastic matrix. According to (U.1) for every eigenvalue λ' of P there exist $\lambda^{(i)} \in \Lambda_i$, $i = 1 \dots m$, with

$$\lambda' = 1 + \frac{1}{q}\sum_{i=1}^m \lambda^{(i)} = \frac{1}{q}\sum_{i=1}^m q_i + \lambda^{(i)}, \tag{2.16}$$

where the last term is obtained by obeying $q = q_1 + \cdots + q_m$. λ' is a complex number, and hence application of the triangle equality yields

$$|\lambda'| = |1 + \frac{1}{q}\sum_{i=1}^{m} \lambda^{(i)}| \leq \frac{1}{q}\sum_{i=1}^{m} |q_i + \lambda^{(i)}|. \tag{2.17}$$

(U.4) and (U.6) state that the sum on the right-hand side becomes maximal if and only if all of the $\lambda^{(i)}$ equal 0. In this case both $|\lambda'|$ and the sum on the right-hand side become 1.

Hence in the representation (2.16) of the eigenvalue λ_2 of P in terms of eigenvalues $\lambda^{(i)}$, $i = 1 \ldots m$, at least one of the $\lambda^{(i)}$ must not equal 0. Clearly the maximum value of $\frac{1}{q}\sum_{i=1}^{m} |q_i + \lambda^{(i)}|$, where at least one of the $\lambda^{(i)}$ not equal 0, is given by (2.15).

□

2.1.2 The Mean Time to Absorption and Related Quantities

Two quantities of U will be of interest to us throughout the rest of this chapter: the mean time to absorption and the mean time spent in some set A before absorption. Obviously, the latter quantity is a fraction of the mean time to absorption.

At first we wish to give some insight into classical approaches to compute these two quantities.

The Mean Time to Absorption: A Classical Approach. Let H be a random variable for the time until absorption of the CTMC U. Classic approaches to compute the mean time $\mathbb{E}[H]$ rely on the fact that H is a phase-type distributed random variable whose generator matrix can be written in the form

$$Q^\circ = \begin{pmatrix} T & T_0 \\ 0 & 0 \end{pmatrix}, \tag{2.18}$$

where Q° results from Q be rearranging the states such that the block T contains the transition rates between the transient states and T_0 contains the rates from transient to absorbing states.

Let $\alpha = (\alpha_1, \ldots, \alpha_{dim(T)})$ be the sub-vector of the starting distribution of U, which contains the initial probabilities of the transient states. Then the phase-type distribution can be characterised by its representation (α, T). It is well known that the $n-$th moment of a phase-type distributed random variable is given by (see e.g. [Neut81])

$$\mathbb{E}[H^n] = (-1)^n n! \alpha T^{-n} \mathbb{1}, \tag{2.19}$$

where $\mathbb{1}$ is a column vector of size $dim(T)$ consisting of ones. In particular, the mean hitting time is given by

$$\mathbb{E}[H] = -1 \alpha T^{-1} \mathbb{1}, \tag{2.20}$$

which could be solved by an explicit matrix inversion. Alternatively, with the solution of $xT = \alpha$, one obtains $\mathbb{E}[H] = -x\mathbb{1}$. Since the state space of U, and hence also the dimension of the matrix T, grows exponentially in the number m of marginal CTMCs, the computation of $\mathbb{E}[H]$ according to (2.20) would in general only be feasible for small values of m.

Fractions of the Mean Time to Absorption: A Classical Approach. From the mean time to absorption, which can also be seen as the mean time the CTMC U spends in any state before absorption, we turn to the mean time which U spends in some set A before absorption. With respect to requirements needed later on in this work, we restrict the set A to possess the structure

$$A = A_1 \times A_2 \times \cdots \times A_m, \tag{2.21}$$

where $A_i \subseteq E_i$, $i = 1 \ldots m$. We impose the restriction that A does not contain absorbing states, i.e. $A \cap S = \emptyset$, or equivalently $\exists i \in \{1, \ldots, m\} : A_i \cap S_i = \emptyset$.

Let H_A be a random variable for the time that U spends in set A before absorption and let π_A be the relative amount of time that U spends in A before absorption. It is clear that the mean time $\mathbb{E}[H_A]$ is given by

$$\mathbb{E}[H_A] = \pi_A \mathbb{E}[H]. \tag{2.22}$$

Now, we concentrate on the computation of π_A. First, we assume that U possesses exactly one starting state. W.l.o.g. assume that the first row and first column of Q° correspond to that starting state. Now, we construct a regenerative process $\mathfrak{U}$, where one regeneration cycle of that new process is described by the absorbing Markov chain U, i.e. if U becomes absorbed, the process $\mathfrak{U}$ steps into a new regeneration cycle. In other words, we manipulate U such that each transition which would lead to an absorbing state is directed to to the starting state instead – this new process we call $\mathfrak{U}$.

Of course, the steady state probability $\mathbb{P}(\mathfrak{U} \in A)$ equals the fraction of time spent in A during one regeneration cycle, and hence we have $\mathbb{P}(\mathfrak{U} \in A) = \pi_A$. More formally, if $\mathfrak{T}$ is the generator matrix[2] of $\mathfrak{U}$, then

$$\pi_A = y(A), \qquad \text{where } \; y\mathfrak{T} = 0, \;\; \| y \|_1 = 1. \tag{2.23}$$

The Mean Time to Absorption: Uniformisation. An alternative approach to compute the mean time to absorption is based on the uniformisation method. The starting point is the following representation of the mean time to absorption

$$\mathbb{E}[H] = \int_0^\infty p(t)(E \setminus S)dt. \tag{2.24}$$

Employing equation (2.4) we obtain

$$\begin{aligned} \mathbb{E}[H] &= \int_0^\infty \sum_{n=0}^\infty \frac{(qt)^n}{n!} e^{-qt} \nu(n)(E \setminus S)dt && (2.25)\\ &= \frac{1}{q} \sum_{n=0}^\infty \nu(n)(E \setminus S) && (2.26)\\ &= \frac{1}{q} \sum_{n=0}^\infty 1 - \nu(n)(S), && (2.27) \end{aligned}$$

where $\nu(0) = p(0)$ and $\nu(n) = p(0)P^n = \nu(n-1)P$, for $n \geq 1$.

For practical computations it is necessary to truncate this sum after some index N, which introduces the absolute error

$$err = \frac{1}{q} \sum_{n=N+1}^\infty 1 - \nu(n)(S). \tag{2.28}$$

In section 2.1.1 we saw that $1 - \nu(k)(S)$ can be bounded by $c|\lambda_2|^k$, for all $k > K$, where λ_2 is the second largest eigenvalue in modulus of P and c is a suited positive constant. Hence, if

[2] Note that $\mathfrak{T}$ can easily be derived from the matrices T and T_0.

c and λ_2 are known the error can be bounded by

$$err < \frac{1}{q} \sum_{n=N+1}^{\infty} c|\lambda_2|^n = \frac{c}{q} \frac{|\lambda_2|^{N+1}}{1-|\lambda_2|}. \tag{2.29}$$

Fractions of the Mean Time to Absorption: Uniformisation. The fraction of time spent in the set A before absorption is given by

$$\mathbb{E}[H_A] = \int_0^\infty p(t)(A)dt, \quad \text{for } A \cap S = \emptyset \tag{2.30}$$

In analogy to the transformations concerning the mean time to absorption in the preceding paragraph, we at first adopt the notation $\nu(n) = p(0)P^n$, for $n \geq 0$, and finally obtain, for $A \cap S = \emptyset$,

$$\mathbb{E}[H_A] = \int_0^\infty \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{-qt} \nu(n)(A)dt = \frac{1}{q} \sum_{n=0}^{\infty} \nu(n)(A). \tag{2.31}$$

Since A contains no absorbing states, we have $\nu(n)(A) \leq \nu(n)(E \setminus S) = 1 - \nu(n)(S)$, and hence

$$\frac{1}{q} \sum_{n=N+1}^{\infty} \nu(n)(A) \leq \frac{c}{q} \frac{|\lambda_2|^{N+1}}{1-|\lambda_2|}. \tag{2.32}$$

2.1.3 Outline

A straight forward way to evaluate the formulas

$$\mathbb{E}[H] = \frac{1}{q} \sum_{n=0}^{\infty} 1 - \nu(n)(S), \tag{2.33}$$

$$\mathbb{E}[H_A] = \frac{1}{q} \sum_{n=0}^{\infty} \nu(n)(A) \tag{2.34}$$

would be to compute the distributions $\nu(n)$ up to some truncation index N, where $\nu(n)$ could be obtained by exploiting the relation $\nu(n) = \nu(0)P^n = \nu(n-1)P$, and afterwards compute the above sums. But since the state space of $U = (U_1, \ldots, U_m)$ grows exponentially in the number m of marginal CTMCs, the dimension of P, as well as the size of the distribution vector $\nu(n)$, is exponential in m. Thus, computing $\mathbb{E}[H]$ and $\mathbb{E}[H_A]$ by the procedure sketched above is only feasible for small values of m.

The next section 2.2 deals with finding a different method to compute the quantities $\nu(n)(A)$ and $\nu(n)(S)$ which circumvents the state space explosion problem. The idea is to express $\nu(n)(A)$ and $\nu(n)(S)$ by means of akin marginal quantities $\nu_i(k)(A_i)$ and $\nu_i(k)(S_i)$, $k = 0 \ldots n$, $i = 1 \ldots m$, which are gained from computations on the marginal chains only.

In section 2.3 we introduce a new method to compute $\mathbb{E}[H]$ and $\mathbb{E}[H_A]$. This method follows formulas (2.33) and (2.34), but employs a special technique for the computation of the $\nu(n)(A)$ and $\nu(n)(S)$, $n = 0 \ldots N$.

2.2 A Non-Product Form Representation for Transient State Probabilities

Let $\mathsf{A} \subseteq E$ be any subset of the state space of the joint CTMC U, with

$$\mathsf{A} = \times_{i=1}^{m} \mathsf{A}_i, \tag{2.35}$$

where $\mathsf{A}_i \subseteq E_i$, $i = 1 \ldots m$.

2.2.1 Transient State Probabilities in Product Form

Transient Probabilities of the Marginal CTMCs. Let $p_i(t) = (p_i(t)(u))_{u \in E_i}$ be the transient probability distribution of the chain U_i at time t. For a given initial distribution $p_i(0)$, some $q_i \geq \max_{j \in E_i}\{|Q_i(j,j)|\}$ and with

$$P_i := I + 1/q_i Q_i, \tag{2.36}$$

the probability $p_i(t)(\mathsf{A}_i)$ can be expressed as the series

$$p_i(t)(\mathsf{A}_i) = \sum_{k=0}^{\infty} \frac{(q_i t)^k}{k!} e^{-q_i t} \nu_i(k)[\mathsf{A}_i], \tag{2.37}$$

where $\nu_i(0) = p_i(0)$ and $\nu_i(k+1) = \nu_i(k) P_i$, for $k \geq 0$. In (2.37) we use the convention

$$\nu_i[\mathsf{A}] = (\nu_i(k)[\mathsf{A}_i])_{k \in \mathbb{N}_0} := (\nu_i(k)(\mathsf{A}_i))_{k \in \mathbb{N}_0}, \quad \text{for } i = 1 \ldots m, \tag{2.38}$$

i.e. we interpret $\nu_i[\mathsf{A}_i]$ as a function of n.

Transient Probabilities of the Joint CTMC U. For some $q \geq \max_{j \in E}\{|Q(j,j)|\}$, define $P = I + \frac{1}{q} Q$. Let $\nu(0) = p(0)$ be the initial distribution of U and $\nu(n) = \nu(0) P^n = \nu(n-1) P$, for $n \geq 1$.

Analogously to the preceding paragraph, for $\nu(n)(\mathsf{A})$ we subsequently write $\nu(n)[\mathsf{A}]$ to stress the fact that A is fixed and $\nu[\mathsf{A}]$ can be seen as a function of n.

On the one hand, the transient probability $p(t)(\mathsf{A})$, that U is in A at time t, can be obtained by the uniformisation equation

$$p(t)(\mathsf{A}) = \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{-qt} \nu(n)[\mathsf{A}]. \tag{2.39}$$

On the other hand, by independence of the marginal CTMCs U_i, $i = 1 \ldots m$, $p(t)(\mathsf{A})$ possesses the product form

$$p(t)(\mathsf{A}) = p_1(t)(\mathsf{A}_1) \cdot p_2(t)(\mathsf{A}_2) \cdots p_m(t)(\mathsf{A}_m), \tag{2.40}$$

where $p_i(t)(\mathsf{A}_i) = \mathbb{P}(U_i(t) \in \mathsf{A}_i)$, $i = 1 \ldots m$.

Although, this product-form relation exists for transient probabilities, a similar product-form result for the cumulative measure $\int_0^\infty p(t)(\mathsf{A}) dt$ is not available.

The remainder of this section deals with a reformulation of (2.40). Based on this reformulation, we will be able to compute cumulative measures of the above kind by operating on the marginal CTMCs U_i, $i = 1 \ldots m$, only.

2.2.2 Transient State Probabilities in Non-Product Form

First recall that $q_i \geq \max_{j \in E_i}\{|Q_i(j,j)|\}$ and $q \geq \max_{j\in E}\{|Q(j,j)|\}$. Since $Q = \oplus_{i=1}^m Q_i$, we have

$$\max_{j\in E}\{|Q(j,j)|\} = \sum_{i=1}^{m} \max_{j\in E_i}\{|Q_i(j,j)|\}. \tag{2.41}$$

Hence for valid uniformisation rates q_i of U_i, $i = 1\dots m$, $\sum_i q_i$ is a valid uniformisation rate of U. Vice versa, for every valid uniformisation rate q of U, there exist valid uniformisation rates q_i of U_i, $i = 1\dots m$, with $q = \sum_i q_i$.

With the substitution

$$q := q_1 + q_2 + \cdots + q_m \tag{2.42}$$

and equations (2.40) and (2.37) the following equalities are obtained

$$\begin{aligned}
p(t)(\mathsf{A}) &= \left(\sum_{n_1=0}^{\infty} \frac{(q_1 t)^{n_1}}{n_1!} e^{-q_1 t} \nu_1(n_1)[\mathsf{A}_1]\right) \cdots \left(\sum_{n_m=0}^{\infty} \frac{(q_m t)^{n_m}}{n_m!} e^{-q_1 t} \nu_m(n_m)[\mathsf{A}_m]\right) && (2.43)\\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \left(t^{(n_1+\cdots+n_m)} e^{-qt} \prod_{i=1}^{m} \frac{q_i^{n_i}}{n_i!} \nu_i(n_i)[\mathsf{A}_i] \right) && (2.44)\\
&= \sum_{n=0}^{\infty} t^n e^{-qt} \sum_{n_1+\cdots+n_m=n} \prod_{i=1}^{m} \frac{q_i^{n_i}}{n_i!} \nu_i(n_i)[\mathsf{A}_i] && (2.45)\\
&= \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{-qt} \sum_{n_1+\cdots+n_m=n} \left[n! \prod_{i=1}^{m} \frac{q_i^{n_i}}{n_i! q^{n_i}} \right] \prod_{i=1}^{m} \nu_i(n_i)[\mathsf{A}_i]. && (2.46)
\end{aligned}$$

Since $n_1 + \cdots + n_m = n$, the above term in brackets is a multinomial probability. We denote this term by

$$M(n, n_1, \dots, n_m) := \frac{n!}{\prod_{i=1}^{m} n_i!} \prod_{i=1}^{m} \left(\frac{q_i}{q}\right)^{n_i} \tag{2.47}$$

and write

$$p(t)(\mathsf{A}) = \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{-qt} \sum_{n_1+\cdots+n_m=n} M(n, n_1, \dots, n_m) \prod_{i=1}^{m} \nu_i(n_i)[\mathsf{A}_i]. \tag{2.48}$$

Comparing (2.48) to (2.39), it might be supposed that the inner sum equals $\nu(n)[\mathsf{A}]$ which is the probability that U is in $\mathsf{A} = (\mathsf{A}_1, \dots, \mathsf{A}_m)$ at the $n-$th step. That this is indeed the case is proved in the following corollary.

Corollary 3. *The following assertion holds.*

$$\nu(n)[\mathsf{A}] = \sum_{n_1+\cdots+n_m=n} M(n, n_1, \dots, n_m) \prod_{i=1}^{m} \nu_i(n_i)[\mathsf{A}_i]. \tag{2.49}$$

Proof. We provide a probabilistic proof. At first, let us recall that the marginal processes U_i, $i = 1\dots m$, are uniformised with rate q_i, i.e. the sequence of steps of U_i forms a poisson stream with rate q_i. Then, the sequence of steps of the joint process U is the superposition of m poisson streams, and hence its rate is given by $q = q_1 + \cdots + q_m$.

Let $\#(U)$ be the number of steps of U , and accordingly let $\#(U_i)$ be the number of steps of U_i. Then, the probability $\nu(n)[A]$ that U is in A in the $n-$th step is given by

$$\begin{aligned}
\nu(n)[A] &= \mathbb{P}(U \in A | \#(U) = n) && (2.50)\\
&= \mathbb{P}(U \in A | \#(U_1) + \cdots + \#(U_m) = n) && (2.51)\\
&= \sum_{n_1,\ldots,n_m \geq 0} \mathbb{P}(\#(U_1) = n_1, \ldots, \#(U_m) = n_m | \sum_i n_i = n) \cdot && (2.52)\\
&\qquad \mathbb{P}(U \in A | \#(U_1) = n_1, \ldots, \#(U_m) = n_m, \sum_i n_i = n) && (2.53)
\end{aligned}$$

Expressing $\{U \in A\}$ in the second term by means of marginal quantities and with the insight that the condition $\{\sum_i n_i = n\}$ in the second term is unnecessary, we obtain

$$\begin{aligned}
\nu(n)[A] &= \sum_{n_1,\ldots,n_m \geq 0} \mathbb{P}(\#(U_1) = n_1, \ldots, \#(U_m) = n_m | \sum_i n_i = n) \cdot && (2.54)\\
&\qquad \mathbb{P}(U_1 \in A_1, \ldots, U_m \in A_m | \#(U_1) = n_1, \ldots, \#(U_m) = n_m) && (2.55)\\
&= \sum_{n_1,\ldots,n_m \geq 0} \mathbb{P}(\#(U_1) = n_1, \ldots, \#(U_m) = n_m | \sum_i n_i = n) \prod_i \nu_i(n_i)[A_i] && (2.56)
\end{aligned}$$

The last line results from the independence of the marginal processes U_i, $i = 1 \ldots m$, and the fact that $\nu_i(n_i)[A_i] = \mathbb{P}(U_i \in A_i | \#(U_i) = n_i)$. Now, we realise that due to the condition $\{\sum_i n_i = n\}$ we only need to sum over values of the n_i, $i = 1 \ldots m$, which sum up to n.

$$\begin{aligned}
\nu(n)[A] &= \sum_{n_1+\cdots+n_m = n} \mathbb{P}(\#(U_1) = n_1, \ldots, \#(U_m) = n_m | \#(U) = n) \prod_i \nu_i(n_i)[A_i] && (2.57)\\
&= \sum_{n_1+\cdots+n_m = n} \frac{\prod_i \frac{q_i^{n_i}}{n_i!} e^{-q_i}}{\frac{q^n}{n!} e^{-q}} \prod_i \nu_i(n_i)[A_i] && (2.58)\\
&= \sum_{n_1+\cdots+n_m = n} \frac{n!}{\prod_i n_i!} \prod_i \left(\frac{q_i}{q}\right)^{n_i} \prod_i \nu_i(n_i)[A_i] && (2.59)
\end{aligned}$$

The last two lines make use of the fact that $\#(U_i)$ is poisson distributed with rate q_i, $i = 1 \ldots m$, and $\#(U)$ is poisson distributed with rate $q = q_1 + \cdots + q_m$. □

The interpretation of the statement of corollary 3 is that, in order to determine $\nu(n)[A]$, we consider all possible combinations of the number of marginal steps, which sum up to n. This is due to the fact that the sum of the marginal steps is the number of steps of the joint process U. Given some combination of marginal steps $(n_1, \ldots, n_m)$ we can immediately state that the conditional probability of being in A is given by $\prod_i \nu_i(n_i)[A_i]$. As it turns out, the probability that a certain combination of marginal steps occurs is a multinomial probability. Thus, $\nu(n)[A]$ is obtained by deconditioning $\prod_i \nu_i(n_i)[A_i]$ from $(n_1, \ldots, n_m)$.

The next section reformulates (2.49) as a convolution-like expression, which can be used to compute $\nu(n)[A]$ in an iterative way.

2.2.3 Transient Probabilities and the $\star$-Convolution

Definition 4. For two discrete functions $f, g : \mathbb{N}_0 \to \mathbb{R}_0$ and constant values $c_f, c_g \in \mathbb{R}_{>0}$ assigned to these functions, define

$$(f \star g)(n) = \sum_{k=0}^{n} \binom{n}{k} \frac{c_f^k c_g^{n-k}}{(c_f + c_g)^n} f(k) g(n-k) \quad (2.60)$$

$$\text{and } c_{f\star g} = c_f + c_g. \quad (2.61)$$

The proof of the following corollary is found in the appendix A.1.

Corollary 5. *The $\star-$operator is commutative and associative.*

Theorem 6. *With the functions $\nu_i[\mathsf{A}_i], 1 \leq i \leq m$, and the values $c_{\nu_i[\mathsf{A}_i]} := q_i$ assigned to them, the following assertion holds*

$$(\nu_1[\mathsf{A}_1] \star \nu_2[\mathsf{A}_2] \star \cdots \star \nu_m[\mathsf{A}_m])(n) = \sum_{n_1+\cdots+n_m=n} M(n, n_1, \ldots, n_m) \prod_{i=1}^{m} \nu_i(n_i)[\mathsf{A}_i]. \quad (2.62)$$

Proof.

$$\sum_{n_1+\cdots+n_m=n} M(n, n_1, \ldots, n_m) \prod_{i=1}^{m} \nu_i(n_i)[\mathsf{A}_i] = n! \sum_{n_1+\cdots+n_m=n} \prod_{i=1}^{m} \underbrace{\frac{q_i^{n_i}}{n_i! q^{n_i}} \nu_i(n_i)[\mathsf{A}_i]}_{a_i(n_i)} \quad (2.63)$$

$$= n!(a_1 * \cdots * a_m)(n), \quad (2.64)$$

where $a_i(n_i) = \frac{q_i^{n_i}}{n_i! q^{n_i}} \nu_i(n_i)[\mathsf{A}_i]$ and $*$ is the discrete convolution operator.

Now, it is sufficient to show that

$$(\nu_1[\mathsf{A}_1] \star \nu_2[\mathsf{A}_2] \star \cdots \star \nu_m[\mathsf{A}_m])(n) = n!(a_1 * a_2 * \cdots * a_m)(n). \quad (2.65)$$

Bearing in mind that $q = q_1 + \cdots + q_m$ it suffices to show that for every $\ell \leq m$ the relation

$$\frac{(q_1 + \cdots + q_\ell)^n}{q^n} (\nu_1[\mathsf{A}_1] \star \cdots \star \nu_\ell[\mathsf{A}_\ell])(n) = n!(a_1 * \cdots a_\ell)(n) \quad (2.66)$$

holds. We proof this by induction over ℓ. For $\ell = 1$ the assertion is trivially true and for $\ell = 2$ we have

$$(\nu_1[\mathsf{A}_1] \star \nu_2[\mathsf{A}_2])(n) = \sum_{k=0}^{n} \binom{n}{k} \frac{q_1^k q_2^{n-k}}{(q_1 + q_2)^n} \nu_1(k)[\mathsf{A}_1] \nu_2(n-k)[\mathsf{A}_2] \quad (2.67)$$

$$= n! \frac{q^n}{(q_1 + q_2)^n} (a_1 * a_2)(n), \quad (2.68)$$

where we multiplied the sum with $\frac{q^n}{q^n}$, in order to obtain the last line. This provides us with a valid induction hypothesis. For the induction step we obtain

$$(\nu_1[A_1]\star \cdots \star\nu_{\ell+1}[A_{\ell+1}])(n) = \tag{2.69}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(q_1+\cdots+q_\ell)^k q_{\ell+1}^{n-k}}{(q_1+\cdots+q_{\ell+1})^n} \underbrace{(\nu_1[A_1]\star\cdots\star\nu_\ell[A_\ell])(k)}_{\substack{=k!\frac{q^k}{(q_1+\cdots+q_\ell)^k}(a_1*\cdots*a_\ell)(k) \\ per\ induction\ hypothesis}} \nu_{\ell+1}(n-k)[A_{\ell+1}] \tag{2.70}$$

$$= \sum_{k=0}^{n} k!\binom{n}{k} \frac{q_{\ell+1}^{n-k}}{(q_1+\cdots+q_{\ell+1})^n} q^k (a_1*\cdots*a_\ell)(k)\nu_{\ell+1}(n-k)[A_{\ell+1}] \tag{2.71}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{q_{\ell+1}^{n-k}}{(q_1+\cdots+q_{\ell+1})^n} \frac{q^n}{q^{n-k}} (a_1*\cdots*a_\ell)(k)\nu_{\ell+1}(n-k)[A_{\ell+1}] \tag{2.72}$$

$$= n!\frac{q^n}{(q_1+\cdots+q_{\ell+1})^n} \sum_{k=0}^{n} \frac{q_{\ell+1}^{n-k}}{(n-k)!q^{n-k}} (a_1*\cdots*a_\ell)(k)\nu_{\ell+1}(n-k)[A_{\ell+1}] \tag{2.73}$$

$$= n!\frac{q^n}{(q_1+\cdots+q_{\ell+1})^n} (a_1*\cdots*a_{\ell+1})(n) \tag{2.74}$$

□

The following theorem summarises corollary 3 and theorem 6.

Theorem 7. *The following is true:*

$$\nu(n)[A] = (\nu_1[A_1]\star\nu_2[A_2]\star\cdots\star\nu_m[A_m])(n) = \sum_{n_1+\cdots+n_m=n} M(n,n_1,\ldots,n_m)\prod_{i=1}^{m}\nu_i(n_i)[A_i]. \tag{2.75}$$

Proof. Corollary 3 and theorem 6. □

To conclude this subsection, we take up equation (2.48) and in consideration of theorem 7 obtain

$$p(t)[A] = \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{-qt} (\nu_1[A_1]\star\nu_2[A_2]\star\cdots\star\nu_m[A_m])(n). \tag{2.76}$$

2.2.4 Cumulative Measures by Means of the $\star$-Convolution

We have seen that the transient probability $\nu(n)[A]$ can be related to the (marginal) functions $\nu_i[A_i]$, $i=1\ldots m$, via the $\star$-convolution.

As a direct consequence, the cumulative measures

$$\mathbb{E}[H] = \int_0^\infty 1-p(t)(S)dt \quad \text{and} \quad \mathbb{E}[H_A] = \int_0^\infty p(t)(A)dt, \quad \text{for } A\cap S=\emptyset \tag{2.77}$$

can also be expressed by means of the $\star$-convolution.

Theorem 8. *For $A_i \subseteq E_i$, $i=1\ldots m$, $A = \times_{i=1}^m A_i$ and $A\cap S=\emptyset$, the expected time $\mathbb{E}[H_A]$ which U spends in A before absorption is given by*

$$\mathbb{E}[H_A] = \frac{1}{q}\sum_{n=0}^{\infty}(\nu_1[A_1]\star\nu_2[A_2]\star\cdots\star\nu_m[A_m])(n). \tag{2.78}$$

Proof. The assertion follows from equation (2.31) in combination with theorem 7. ☐

Theorem 9. *The expected time until absorption* $\mathbb{E}[H]$ *of* U *is given by*

$$\mathbb{E}[H] = \frac{1}{q}\sum_{n=0}^{\infty} 1 - (\nu_1[S_1] \star \nu_2[S_2] \star \cdots \star \nu_m[S_m])(n). \tag{2.79}$$

Proof. The assertion follows from equation (2.27) in combination with theorem 7.

☐

2.3 Algorithms for $\mathbb{E}[H]$ and $\mathbb{E}[H_A]$

This section aims at giving all means needed for an implementation of a programme which computes $\mathbb{E}[H]$ and $\mathbb{E}[H_A]$.

Therefore, we provide a short compilation of the key results (section 2.3.1) which have been the subject of previous sections, followed by the actual algorithm and a detailed complexity analysis (section 2.3.2).

2.3.1 A Short Summarization

In order to compute the two quantities

$$\mathbb{E}[H] = \frac{1}{q}\sum_{n=0}^{\infty} 1 - \nu(n)[S], \tag{2.80}$$

$$\mathbb{E}[H_A] = \frac{1}{q}\sum_{n=0}^{\infty} \nu(n)[A] \tag{2.81}$$

it is necessary to truncate the sum after a certain index N, i.e. we actually compute

$$\mathbb{E}[H] \approx \frac{1}{q}\sum_{n=0}^{N} 1 - \nu(n)[S], \tag{2.82}$$

$$\mathbb{E}[H_A] \approx \frac{1}{q}\sum_{n=0}^{N} \nu(n)[A]. \tag{2.83}$$

With the second largest eigenvalue in modulus λ_2 of P and a suited constant c the absolute errors can be bounded by

$$\frac{1}{q}\sum_{n=N+1}^{\infty} 1 - \nu(n)[S] < \frac{c}{q}\frac{|\lambda_2|^{N+1}}{1-|\lambda_2|} =: error \tag{2.84}$$

$$\frac{1}{q}\sum_{n=N+1}^{\infty} \nu(n)[A] < \frac{c}{q}\frac{|\lambda_2|^{N+1}}{1-|\lambda_2|} =: error. \tag{2.85}$$

The absolute value of the eigenvalue λ_2 can be computed by exploiting the fact that

$$\lim_{n\to\infty} \frac{1-\nu(n+1)[S]}{1-\nu(n)[S]} = |\lambda_2|.$$

After $|\lambda_2|$ has been determined (or estimated) the constant c can be estimated by considering

$$\lim_{n\to\infty} \frac{1 - \nu(n)[S]}{|\lambda_2|^n} = constant. \tag{2.86}$$

That means we chose some $c > constant$.

2.3.2 Algorithm

We now present a detailed algorithm in pseudo-code (Procedure CM()) which computes approximations of $\mathbb{E}[H_A]$ and $\mathbb{E}[H]$.

The input of this procedure comprises of the generator matrices Q_i, $i = 1 \ldots m$, of the marginal CTMCs, the marginal starting states s'_i, $i = 1 \ldots m$, and an upper error bound ϵ.

The predicate $precision()$ becomes true if the current values of $|\lambda_2|$ and c fulfil a given precision. Although, not explicitly implemented in the algorithm below, it is clear how an implementation of such a predicate could look like. For example, the values in lines 20 and 21 could be stored (for a while), and the relative changes of these values determine the predicate $precision$.

Procedure CM$(Q_1,\dots,Q_m; s'_1,\dots,s'_m; A_1,\dots,A_m;\epsilon)$

/* ---------------- Uniformise all marginal processes ---------------- */

1 **for** $i = 1\dots m$ **do**
2   choose $q_i \geq \max_{j\in E_i}\{|Q_i(j,j)|\}$; /* Uniformisation factor */
3   $P_i = I + \frac{1}{q_i}Q_i$; /* One-step transition matrix */
4   $\nu_i(0)(s'_i) = 1$; /* Initial state */

/* -------------------- Some initial assignments -------------------- */

5 $\nu(0)[S] = (\nu_1[S_1] \star \nu_2[S_2] \star \dots \star \nu_m[S_m])(0)$;
6 $\nu(0)[A] = (\nu_1[A_1] \star \nu_2[A_2] \star \dots \star \nu_m[A_m])(0)$;
7 $q = q_1 + q_2 + \dots + q_m$;
8 $n = 0$;
9 choose $W \geq 1$;
10 $error = 2\epsilon$; /* initially chosen such that $error > \epsilon$ */

/* ---------- Compute marginal distributions and convolution ---------- */

11 **while** *not* $precision()$ *and* $error \geq \epsilon$ **do**
12   **for** $i = 1\dots m$ **do**
13   **for** $k = n+1\dots n+W$ **do**
14       $\nu_i(k) = \nu_i(k-1)P_i$; /* distribution at the k-th step */
15   compute $\nu_i(k)[A_i]$; /* the aggregated state probabilities */
16   compute $\nu_i(k)[S_i]$; /* $\nu_i(\cdot)[A_i]$ and $\nu_i(\cdot)[S_i]$ are stored permanently */
17   **for** $k = n+1\dots n+W$ **do**
18     $\nu(k)[S] = (\nu_1[S_1] \star \nu_2[S_2] \star \dots \star \nu_m[S_m])(k)$;
19     $\nu(k)[A] = (\nu_1[A_1] \star \nu_2[A_2] \star \dots \star \nu_m[A_m])(k)$;
20     $|\lambda_2| = \frac{1-\nu(k)[S]}{1-\nu(k-1)[S]}$; /* Approximation of $|\lambda_2|$ */
21     $c = \frac{1-\nu(k)[S]}{|\lambda_2|^k}$;
22   $n = n + W$;
23   $error = \frac{c}{q}\frac{|\lambda_2|^{n+1}}{1-|\lambda_2|}$;

/* ------------------- Compute cumulative measures ------------------- */

24 $N = n$;
25 $a = \sum_{k=0}^{N} \frac{1}{q}\nu(k)[A]$; /* Approximation to $\mathbb{E}[H_A]$ */
26 $b = \sum_{k=0}^{N} \frac{1}{q}(1-\nu(k)[S])$; /* Approximation to $\mathbb{E}[H]$ */

In lines 1 to 4 the m marginal CTMCs are uniformised. P_i is the one-step transition matrix resulting from the generator Q_i. Initially, all probability mass of the i–th CTMC is gathered in the state s'_i.

In lines 5 to 10 some initial computations are carried out.

The interesting work of the algorithm is carried out in the while loop beginning in line 11. This while-loop is iterated through until the error of the desired quantities and the estimated parameters $|\lambda_2|$ and c that are used to compute the error fulfil a given precision. During each pass of the while-loop W more elements of the functions $\nu[A]$ and $\nu[S]$ are computed, i.e. in the first pass $\nu(1)[A],\dots,\nu(W)[A]$ are computed, in the second pass $\nu(W+1)[A],\dots,\nu(2W)[A]$, and so on ($\nu(\cdot)[S]$ accordingly). Note, that the values of $\nu(\ell)[A]$ and $\nu(\ell)[S]$ can indeed be computed, since all the values $\nu_i(n)[A_i]$ and $\nu_i(n)[S_i]$, $n = 0\dots\ell$, $i = 1\dots m$, have been